SOLVING THE VISCOUS COMPOSITE CYLINDER PROBLEM BY SOKOLOV'S METHOD

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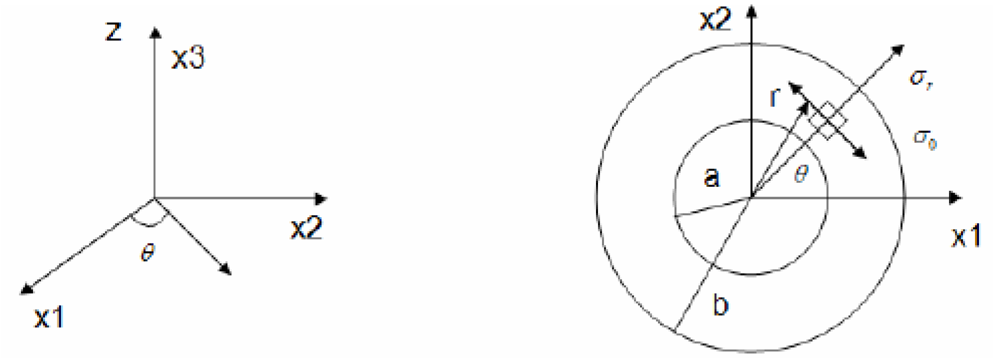
Abstract : *The paper presents some thoughts about the plane strain problem of the viscous orthotropic composite materials cylinder under internal and external pressure with respect to using the average approximating method . To compute the interior stress , from the elastic solution we use the Volterra's principle and Sokolov's method in the corresponding integral equation to find the viscous solution .*

I. The axial symmetric plane strain problem of cylinder :

We examine an orthotropic viscoelastic composite material cylinder which has the horizontal section within limit of 2 circles : r = a , r = b (a < b) .



Choosing the cylindrical coordinates r , θ , z (the axial z is along with the cylinder) . The components of stress and deformation ε_r , ε_θ , σ_r , σ_θ are functions of r , t respectively . The two components of deformation-tensor :

$$\varepsilon_r\ (r,t) = \frac{\partial u(r,t)}{\partial r} \quad ; \quad \varepsilon_\theta(r,t) = \frac{u(r,t)}{r}$$

(1.1)

and the differential equation of equilibrium is :

$$r\frac{\partial \sigma_r(r,t)}{\partial r} + \sigma_r(r,t) - \sigma_\theta(r,t) = 0$$

(1.2)

when t = 0 , boundary conditions : $\sigma_r(a,0) = -P \quad ; \quad \sigma_r(b,0) = -Q$
(1.3)

II . The Volterra Integral equation of the second kind :

The displacement -differential equation of the cylinder in the case of viscoelastic plane-deformation :

$$r^2\left(\frac{\partial^2}{\partial r^2}\mathrm{u}(r,\,t)\right) + r\left(\frac{\partial}{\partial r}\mathrm{u}(r,\,t)\right) - \frac{E_\theta\,\mathrm{u}(r,\,t)}{E_r} = 0$$

(2.1)

The general solution is :
(2.2)

$$\mathrm{u}(r,\,t) := C_1\, r^{\left(\frac{\sqrt{E_\theta}}{\sqrt{E_r}}\right)} + C_2\, r^{\left(-\frac{\sqrt{E_\theta}}{\sqrt{E_r}}\right)}$$

C_1 , C_2 are the arbitrary constants .

The elastic constants in (2.1) will be substituted by operators respectively , from the Volterra's principle we have : ([1])

$$r^2\frac{\partial^2 u(r,t)}{\partial r^2} + r\frac{\partial u(r,t)}{\partial r} - \frac{\hat{E}_\theta}{\hat{E}_r}u(r,t) = 0$$

(2.3)

The equation (2.3) is rewritten as :

$$\frac{\partial^2 u(r,t)}{\partial r^2} + a_1(r)\frac{\partial u(r,t)}{\partial r} + a_2(r)u(r,t) = 0$$

$$a_1(r,t) = \frac{1}{r} \quad , \quad a_2(r,t) = \frac{\hat{E}_\theta}{r^2 \hat{E}_r} = -\frac{m(t)}{r^2}$$

(2.4)

Assume that $u(a,t) = C_0 \quad , \quad u'(a,t) = C_1$ (in [2]) we obtain the solution of (2.4) from :

$$\phi(x,t) + \int_a^x K[(x,y);t]\phi(y,t)dy = f(x,t)$$

(2.5)

The kernel expression of : $K[(x,y);t] = a_1(x,t) + a_2(x,t)\frac{(x-y)}{1!}$

(2.6)

And the formula of *f(x,t)* is : $f(x,t) = -C_0 a_1(x,t) - [C_1 x + C_0]a_2(x,t)$
(2.7)

In this case : $K[(x,y);t] = \frac{1}{x} - \frac{m(x-y)}{x^2}$
(2.8)

$$m := \frac{(100. + t^{(1/10)})\, t^{(2/5)}}{100. + \sqrt{t}}$$

(2.9)

From the experimental test , if we have : $\sigma_\theta = 0 \quad , \quad \varepsilon_r(r,0) = Const$
(2.10)

Then : $\sigma_r(t) = \varepsilon_r . E_r(t)$
(2.11)

From the results of (2.5) we obtain the analytical solution of $u(x,t)$, $\sigma_\theta(a,t)$ and $\sigma_\theta(b,t)$.

III. The approximate solution of the Volterra Integral equation of the second kind :

3.1 The Sokolov's average approximate method :

The convergence speed of calculation process can be increased by Sokolov's method . The basic contents of this method is described as following :

From the integral equation : $$u(t) = v(t) + \lambda\int_a^b K(t,\tau)u(\tau)d\tau$$ (3.1)

The n-order expression of *u(t)* is : $$u_n(t) = v(t) + \lambda\int_a^b K(t,\tau)\{u_n(\tau) + \alpha_n\}d\tau$$

(3.2)

This relation has connection with the adjustment quantity :

$$\alpha_n = \frac{1}{b-a}\int_a^b \Delta_n(t)dt$$

(3.3)

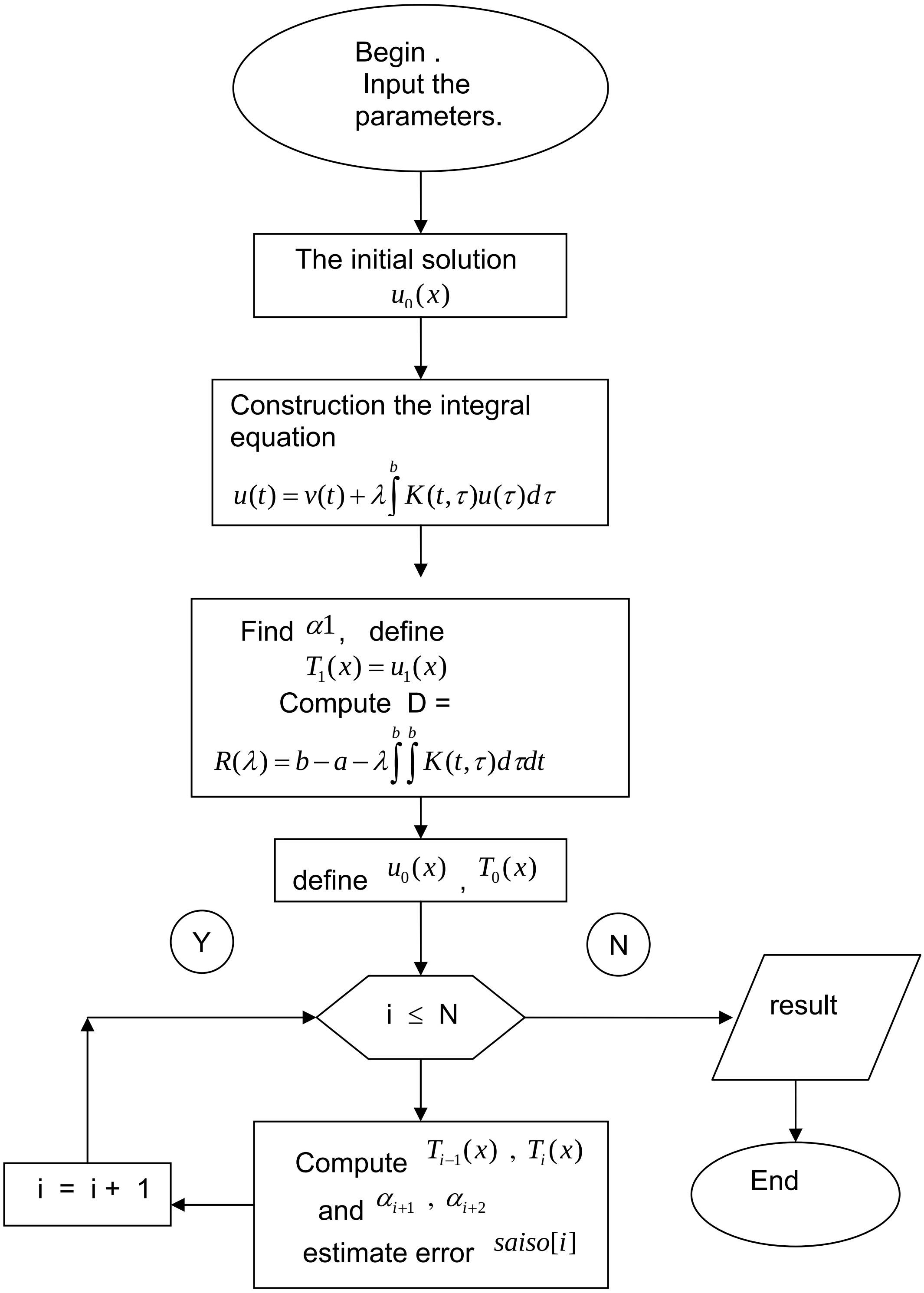
The difference between two results : $$\Delta_n(t) = u_n(t) - u_{n-1}(t)$$ (3.4)

The recurrence relation : $$\alpha_n = \frac{\lambda}{R(\lambda)}\int_a^b\int_a^b K(t,\tau)\{\Delta_{n-1}(\tau) - \alpha_{n-1}\}d\tau dt$$ (3.5)

Note that the convergence -condition of Sokolov's method is $\|L_\lambda\| = \left\|\lambda(I-\lambda K\Pi)^{-1}K(I-\Pi)\right\| < 1$

Here Π is the project operator from the Banach 's space B into its subspace Bo (u $\in$ B) ([2])

3.2 The flow chart of Sokolov's method :



3.3. Problem definition – parameters and the analytical solution :

In the following , modules $E_r(r,t)$ and $E_\theta(r,t)$ are given of the exponential expressions :

a = 1 , b = 2 , $C_0 = 0$, $C_1 = 1$; $E_e = 0.5$, $T_0 = 1$;

1. Parameters Information :

:>m:=(1/x^2)*(100.+(t/To)^(1/10))*(t/To)^(2/5)/(100.+(t/To)^(1/2));To:=1;

We should choose $n := 4$, $n := 5$, $n := 6$

2. Activate the procedure : **> xapxi (m/x, -1 , 1 , 2 , 4 , K) ;**

$K := -\dfrac{m\,(x-y)}{x^2} + \dfrac{1}{x}$; $To := 1$

"--------------------------------KETQUA-------------------------------- "

$$E_\theta(r,t) := \left(\frac{100}{t^{0.1}} + 1\right) Ee$$

$$\sigma_\theta(x,t) := \left(\frac{\ln(x)\,x\,(100.+t^{(1/10)})\,t^{(2/5)}}{100.+\sqrt{t}} - \frac{1.\,(100.+t^{(1/10)})\,t^{(2/5)}\,x}{100.+\sqrt{t}} - 0.002500000000\left(\right.\right.$$

$$\left.\left. + \frac{0.1013313763\ 10^{30}\,(100.+t^{(1/10)})^{12}\,t^{(24/5)}}{(100.+\sqrt{t})^{12}} \ldots \right)\right)\left(\frac{100}{t^{0.1}}+1\right) Ee/x$$

Comment **:** In this procedure note that we should choose the number n of recurrence from 4 up to 6 , so we get the result which is coincide to the solution of Schapery in

[4] . In the comparison with the result in [3] the analytical solution of $\sigma_\theta(a,t)$ and $\sigma_\theta(b,t)$ obtained by Sokolov's method has a better smoothness . Figure 1 and 2 describe

the convergence of $\varepsilon_\theta\ (r,t)$ and $\varepsilon_r\ (r,t)$. Especially , if n > 6 and using Maple version 9.5 it is a waste of time to treat the problem in detail , moreover the accuracy of

solution will be influenced by accumulation of error in programme .

Using Maple version 6.0 the convergence of method can be found correctly when n = 5 .

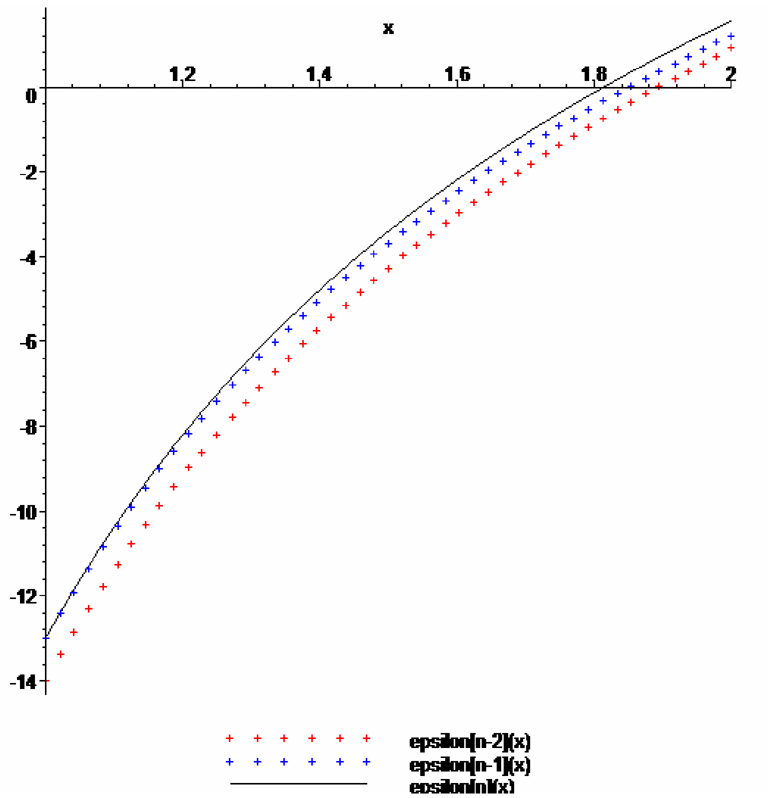


Fig 1 .

graph of strain $\varepsilon_\theta\ (r,t)$

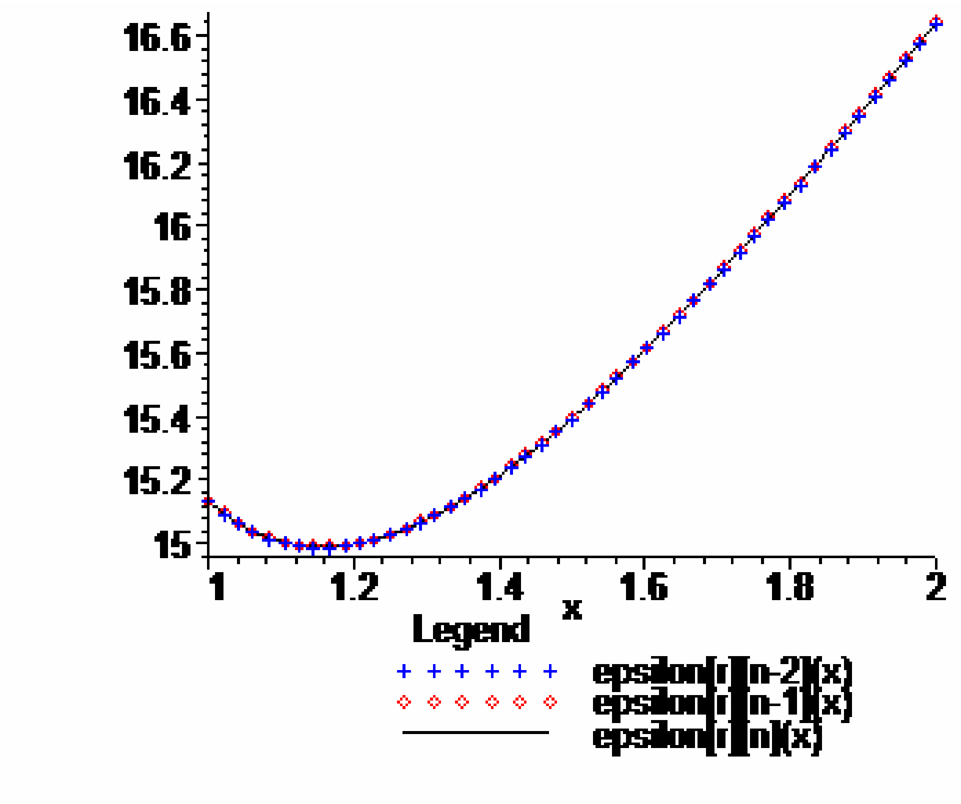


Fig 2 .

convergence of strain $\varepsilon_r\,(r,t)$

Figure 3. and 4. show the manners of the graphs of stress coincide to the work of R.A.Schapery . To obtain these result , the most important work stage is the

transformation differential equation to the correspondent integral equation and setting up the suitable boundary conditions.

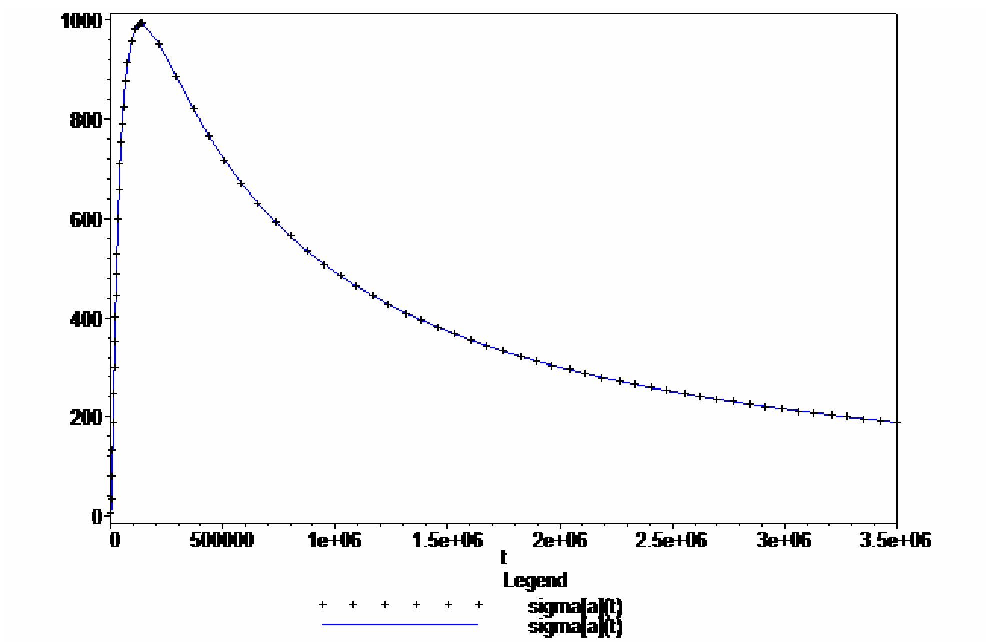


Fig 3 .

graph of stress $\sigma_\theta(a,t)$, *a = 1*

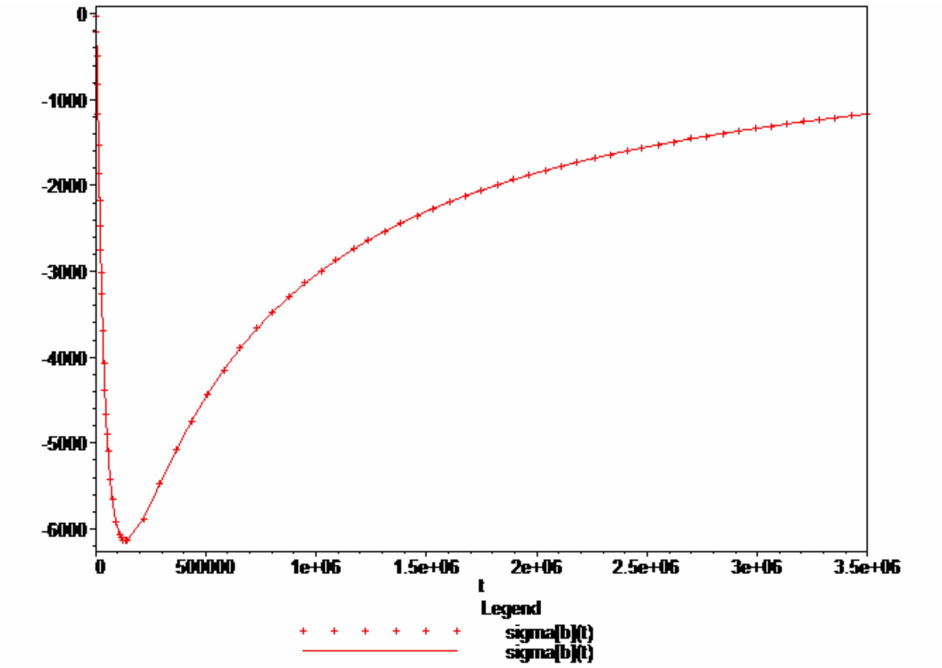
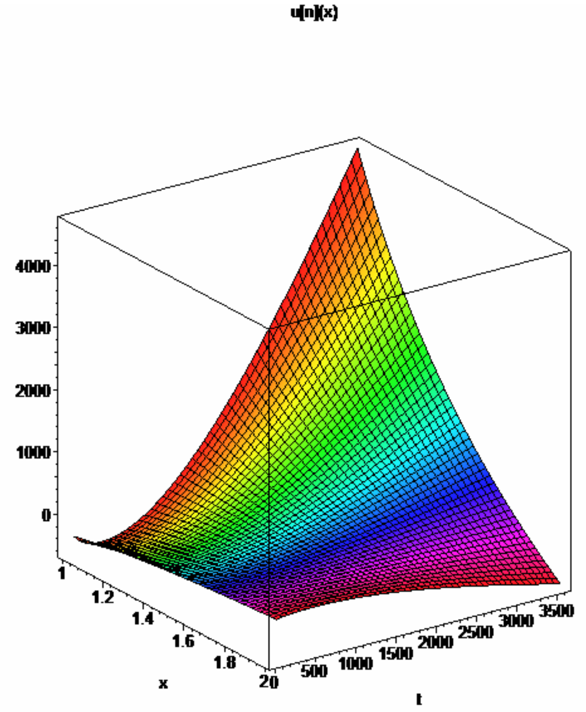


Fig 4 .

graph of stress $\sigma_\theta(b,t)$, *b = 1*

In Figure 5 we notice the schematic representation of the displacement at x = 1 and x = 2 ; the manners of the graphs of strain $\varepsilon_r\,(r,t)$ and convergence process of

$\varepsilon_r\,(r,t)$ according to the order { n -2 , n-1 ,n } can be shown in figure 6 .



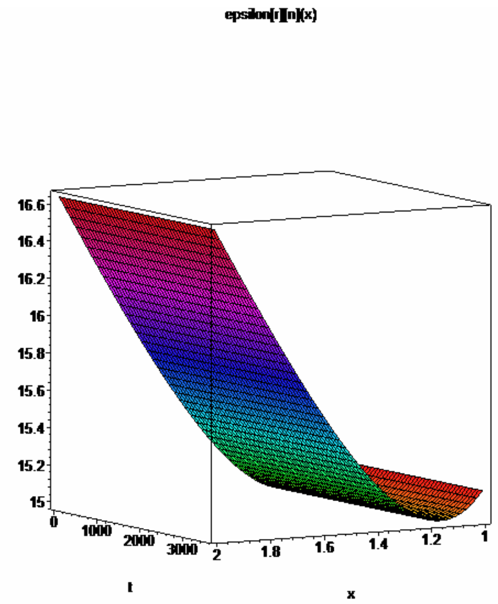
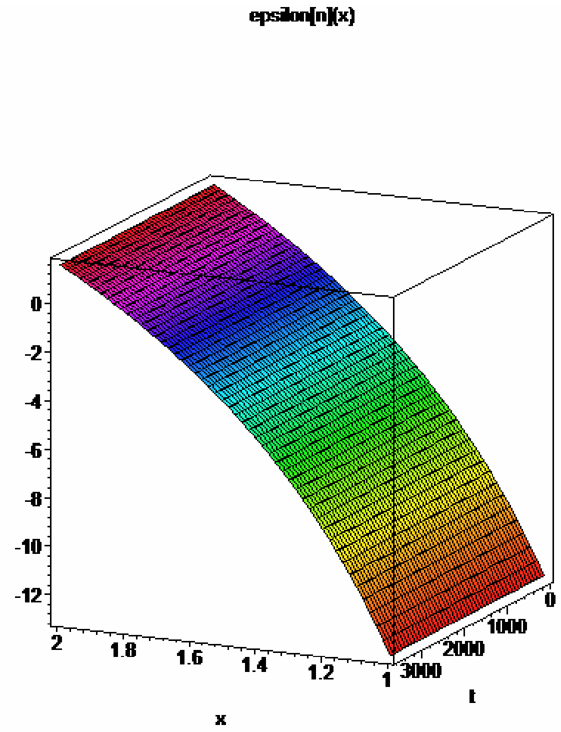


Fig 5 . graph of displacement $u(r,t)$

Fig 6 graph of $\varepsilon_r(r,t)$.



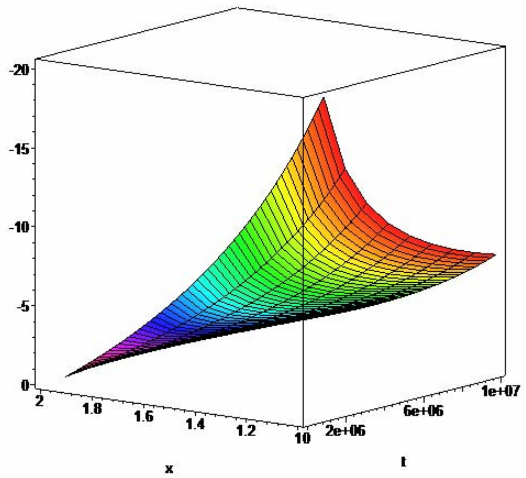


Fig 7 . graph of $\varepsilon_\theta\,(r,t)$.

Fig 8 . graph of $\sigma_\theta(r,t)$.

IV . Conclusion :

The plane strain problem of the viscous orthotropic composite materials cylinder under internal and external pressure can be solved by other approximate methods : direct , collocation , quasi-elastic , Laplace transform inversion

… ([3]) . Almost the algorithm regularly depends on the displacement differential equation and the establishment of procedure are also carried out from it . ***The Average Approximating Method on Functional Adjustment***

Quantity (Sokolov's method) ***and*** ***the Volterra's principle*** *corresponding to the integral equation may be used to find the viscous solution . Moreover this application makes increasing for the convergence speed of the*

solution $u_n(t)$ *. From the first approximation of the solution u , we find the adjustment quantity for the next and so on . Note that we should choose the number n of recurrence and the expressions of* $a_1(x,t)$, $a_2(x,t)$

appropriately to get the analytical result which is satisfied the given boundary conditions of the problem .

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